LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc., DEGREE EXAMINATION - **MATHEMATICS**

SECOND SEMESTER – APRIL 2015

MT 2810– ALGEBRA

Max. : 100 Marks

Answer **ALL** the Questions:

1. (a) If G is a finite group, then prove that $c_a = \dfrac{O(G)}{O\left(N(a)\right)}$; in other words, show that the number of elements conjugate to a in G is the index of the normalizer of a in G.

(OR)

(b) Show that "Conjugacy is an equivalence relation in G." (5)

(c) State and prove Cauchy Theorem.

(OR)

(d) If p is a prime number and p^{α} divides $O(G)$ then prove G has a subgroup of order p^{α}. (15)

2. (a) State and prove the Division Algorithm.

(OR)

(b) If R is an integral domain, then prove that $R[x]$ is an integral domain. (5)

(c) State and prove the Eisenstein criterion.

(OR)

(d) State and prove the "Fundamental theorem on Finitely generated modules over Euclidean rings". (15)

3. (a) If L is a finite extension of K and K is a finite extension of Fthen prove that L is a finite extension of F. Moreover, $[L\colon F] = [L\colon K][K\colon F]$.

(OR)

(b) Let $f(x) \in F[x]$ be of degree $n \geq 1$, prove that there is an extension E of F of degree atmost $n!$ in which $f(x)$ has n roots. (5)

(c) Prove that the element $a \in K$ is algebraic over F iff $F(a)$ is a finite extension of F.

(OR)

(d) If $p(x)$ is a polynomial in $F[x]$ of degree $n \geq 1$ and it is irreducible over F then show that there is an extension E of F such that $[E\colon F] = n$ in which $p(x)$ has the root. (15)

4. (a) Let F_0 be the field of rational numbers and let $K = F_0\left(\sqrt[3]{2}\right) = \left\{a + b\sqrt[3]{2} + c\left(\sqrt[3]{2}\right)^2 | a, b, c \in F_0\right\}$. Find $G(K, F_0)$ and prove that $K_{G(K,F_0)} = K$.

(OR)

(b) If K is a finite extension of F, the $G(K,F)$ is a finite group and its order satisfies $o(G(K,F) \leq [K:F]$. (5)

(c) Let K be a normal extension of F and let H be a subgroup of $G(K,F)$, $K_H = \{x \in K | \sigma(x) = x, \forall\, \sigma \in H\}$ is the fixed field of H then [i] $[K:K_H] = o(H)$ and [ii] $H = G(K:K_H)$. In particular, $H = G(K,F)$ and $[K:F] = o(G(K,F))$.

(OR)

(d) State and prove the fundamental theorem of Galois theory. (15)

5. (a) If the field F has p^m elements then show that F is the splitting field of the polynomial $x^{p^m} - x$.

(OR)

(b) Prove that any two finite fields having the same number of elements are isomorphic.

(5)

(c) Show that S_n is not solvable for $n \geq 5$.

(d) Verify S_3 is solvable. (8+7)

(OR)

(e) State and prove Wedderburn's theorem on finite division rings. (15)